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Non-integrability of SU(2) Yang–Mills and Yang–Mills–Higgs systems

M P Joy and M Sabir

Department of Physics, Cochin University of Science and Technology, Cochin-682 022, India

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Abstract. The spherically symmetric time-dependent SU(2) Yang-Mills equations and Yang-Mills-Higgs equations are shown to be non-integrable by using the Weiss, Tabor and Carnevale method of Painlevé analysis. Reduced equations corresponding to these systems are also found to be non-integrable.

1. Introduction

Recently the question of integrability of non-Abelian gauge fields has attracted wide attention (Nikolaevskii and Schur 1982, 1983, Savvidy 1984, Matinyan *et al* 1986, 1988, Villarroel 1988). Trajectories of integrable systems are regular and do not show any sensitive dependence on initial conditions. But most of the non-linear classical systems are non-integrable and show complicated behaviour known as chaos. It has been shown that chaos can appear in the classical theory of non-Abelian gauge fields also, at least under certain approximations. This is an important result deserving further study in view of the result obtained by Olsen (1982) that the presence of random fields in the vacuum is a necessary and sufficient condition of quark confinement in quantum chromodynamics.

Most studies made so far have confined themselves to the finite-dimensional subsystems depending only on a time variable. Classical Yang-Mills theory depending only on time (YM classical mechanics) has been shown to be non-integrable and chaotic by various techniques (Matinyan *et al* 1981, Nikolaevskii and Schur 1982, 1983, Gorski 1984, Savvidy 1984, Steeb *et al* 1986, Furusawa 1987, Villarroel 1988). However, with regard to the general $3+1$ field systems the situation is not fully understood. By the Painlevé criterion, SU(2) self-dual Yang-Mills equations have been shown to be integrable (Jimbo *et al* 1982, Ward 1984). But such analysis has not been carried out for more general cases. On the other hand, Matinyan *et al* (1986, 1988) have recently shown that a spacetime-dependent spherically symmetric Yang-Mills system can exhibit dynamical chaos. They employed the Fermi-Pasta-Ulam (1955) method in which continuous equations are replaced by a set of discrete equations which are then numerically analysed. But it is well known that the discretisation itself can cause chaotic behaviour. Also, the continuum limit of a discrete model exhibiting chaos can be non-chaotic.

In this work an attempt will be made to clarify the question of non-integrability in the spherically symmetric non-self-dual sector of SU(2) Yang-Mills and Yang-Mills-Higgs theory without introducing discretisation. We apply singular point analysis to

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test the integrability of the PDE as well as of the ODE obtained by symmetry reduction and by other means. Our results show that these systems are generally non-integrable.

The partial differential equations corresponding to the SU(2) theory and the ODE obtained from them are described in section 2. In section 3 we briefly describe the WTC algorithm for singular point analysis. The results are also presented in this section. Section 4 is a summary of results and conclusions.

2. Yang–Mills and Yang–Mills–Higgs systems

The SU(2) Yang-Mills system is described by the Lagrangian

$$\mathscr{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon_{abc}A^b_\mu A^c_\nu$$

$$\mu,\ \nu = 0, 1, 2, 3 \qquad a, b, c = 1, 2, 3.$$

In the spherically symmetric ansatz,

$$A^a_0 = 0 \qquad A^a_i = \frac{1}{g}\,\varepsilon_{ain}\frac{r_n}{r^2}(1 - K(r, t))$$

the equations of motion, $D_\mu F^{\mu\nu a} = 0$ becomes

$$r^2(K_{rr} - K_{tt}) + K(1 - K^2) = 0. \tag{1}$$

Of the static solutions of (1), $K = 0$ is the Wu-Yang monopole solution, $K = -1$ is the vacuum solution and $K = 1$ is gauge equivalent to the vacuum one. These static solutions are all unstable. All solutions except the trivial one $K = \pm 1$ are non-self-dual. A Lie symmetry analysis for this system including a Higgs field was carried out by Babu Joseph and Baby (1986) and from this we infer that (1) admits a similarity variable,

$$\rho = r/(t^2 - r^2) \tag{2}$$

and on substituting (2) in (1) we get the corresponding similarity reduced ODE

$$\rho^2\frac{\mathrm{d}^2K}{\mathrm{d}\rho^2} = K(K^2 - 1). \tag{3}$$

The singularity analysis of this equation is of significance in view of the conjecture by Ablowitz *et al* (1980), that a system of PDE is integrable if the corresponding similarity reduced system of ODE possesses the Painlevé property. It is also known that using an independent variable transformation (Arodz 1983)

$$\kappa = \frac{t - t_0}{r} - 1 \tag{4}$$

the non-linear partial differential equation (1) can be reduced to a second-order non-linear ordinary differential equation,

$$(2 + \kappa)\kappa\frac{\mathrm{d}^2K}{\mathrm{d}\kappa^2} + 2(1 + \kappa)\frac{\mathrm{d}K}{\mathrm{d}\kappa} + K(1 - K^2) = 0. \tag{5}$$

The domain of κ is $-1 \leqslant \kappa < \infty$.

Another system which we analyse is the SU(2) Yang-Mills-Higgs system with Lagrangian density,

$$\mathcal{L} = -\tfrac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu a} + \tfrac{1}{2}D_\mu\phi^a D^\mu\phi^a - V(\phi)$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon_{abc}A^b_\mu A^c_\nu$$

$$D_\mu\phi_a = \partial_\mu\phi_a + g\varepsilon_{abc}A^b_\mu\phi_c$$

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2.$$

The equations of motion are

$$D_\nu F^{\mu\nu a} = -g\varepsilon_{abc}(D^\mu\phi_b)\phi_c \qquad D_\mu D^\mu\phi_a = (m^2 - \lambda\phi^2)\phi_a.$$

Using the time-dependent 't Hooft-Polyakov ansatz (Mecklenberg and O'Brien 1978)

$$A^a_0 = 0 \qquad A^a_i = \frac{1}{g}\,\varepsilon_{ain}r_n\frac{1-K(r,t)}{r^2}$$

$$\phi_a = \frac{1}{g}\,r_a\frac{H(r)}{r^2}$$

the field equations of the SU(2) gauge theory become

$$r^2(K_{rr} - K_{tt}) = K(K^2 - 1 + H^2)$$
$$r^2(H_{rr} - H_{tt}) = H\left(2K^2 - m^2r^2 + \frac{\lambda}{g^2}H^2\right). \tag{6}$$

In the Prasad-Sommerfeld (PS) limit they reduce to

$$r^2(K_{rr} - K_{tt}) = K(K^2 - 1 + H^2) \qquad r^2(H_{rr} - H_{tt}) = 2HK^2. \tag{7}$$

By using the similarity variable in (2), equations in (7) can be reduced to the system of ODE,

$$\rho^2\frac{\mathrm{d}^2K}{\mathrm{d}\rho^2} = K(K^2 - 1 + H^2) \qquad \rho^2\frac{\mathrm{d}^2H}{\mathrm{d}\rho^2} = 2HK^2. \tag{8}$$

By the independent variable transformation (4), the system (7) yields the ODE

$$(2+\kappa)\kappa\frac{\mathrm{d}^2K}{\mathrm{d}\kappa^2} + 2(1+\kappa)\frac{\mathrm{d}K}{\mathrm{d}\kappa} = K(K^2 - 1 + H^2)$$
$$(2+\kappa)\kappa\frac{\mathrm{d}^2H}{\mathrm{d}\kappa^2} + 2(1+\kappa)\frac{\mathrm{d}H}{\mathrm{d}\kappa} = 2HK^2. \tag{9}$$

It is not known whether the transformation (4) is related to any symmetry or invariance of the system or whether there are other ODE which may be obtained from (1) and (7).

3. Singular point analysis and integrability

3.1. Painlevé analysis

There are no general methods available at present to determine the integrability of a dynamical system. Nowadays, the Painlevé property (PP) is widely being made use of in identifying integrable cases. If the solution of a system of ODE in the complex time plane does not have any movable singularities other than poles, it is said to possess the PP. Systems possessing the PP are believed to be integrable. For a system of ordinary differential equations (ODE)

$$\frac{dx_i}{dt} = F_i(x_1, \ldots, x_n) \tag{10}$$

with $i = 1, 2, \ldots, n$, n being the order of the system, to have the PP there must be a Laurent expansion for solutions of the form

$$x_i = \tau^{P_i} \sum_{j=0}^{\infty} a_{ij}\tau^j \tag{11}$$

where $\tau = t - t_0$ and t_0 is the arbitrary pole position. In the series, $n-1$ expansion coefficients are arbitrary. Using the ARS algorithm (Ablowitz, Ramani and Segur 1980), we can test whether a system of ODE satisfies the necessary condition for possession of the PP.

The notion of the Painlevé property originally introduced for ODE has, in recent times, been generalised to PDE as well. However, there exist several alternative approaches to this idea as applied to PDE. The early suggestion of ARS was to attribute the PP to a PDE when all the ODE obtained by symmetry reduction have the PP. A more useful definition in terms of singular manifolds has been given by Weiss *et al* (1983). Ward (1984) has also proposed a generalised definition of the PP. In the WTC method the dependent variables are expanded as a generalised Laurent series about a singular manifold $\phi = 0$ of the form

$$u_i = \phi^{\alpha_i} \sum_{j=0}^{\infty} u_{ij}\phi^j \tag{12}$$

where $u_{i0} \neq 0$, $u_{ij} = u_{ij}(z_1, z_2, \ldots, z_n)$ and $\phi = \phi(z_1, z_2, \ldots, z_n)$ are analytic functions of the indepenent variables $z_1, z_2, \ldots, z_n$. Substituting equation (12) into the equation of motion we get recursion relations between u_{ij}. The three steps of the algorithm are: (i) find the dominant behaviour, (ii) find the resonances and (iii) find the arbitrary expansion coefficients. In the first step we substitute the $j = 0$ term of the series (12) into the system of equations and calculate α_i for which there is a balance of leading terms. For the PP, α_i should be a negative integer. (For the weak PP it can be a rational number as has been shown for ODE by Ramani *et al* (1982).) From this step we can find u_{io} also.

To find resonances, j, which are the powers at which the coefficients of u_{ij} of the term $\phi^{j+\alpha}$ in the expansion (12) is arbitrary, we substitute $u_{ij} = \phi^{\alpha}(u_{i0} + u_{ij}\phi^j)$ into the equations containing leading-order terms only. Then we extract the coefficient $\tilde{Q}(j) = Q(j)u_{ij}$ of the powers $\phi^{j+\alpha-N}$, where N is the order of the equation. Resonances are roots of the equation $Q(j) = 0$. We always find -1 to be a root which corresponds to the arbitrariness of ϕ. To avoid any movable critical manifolds, we require that the remaining roots are non-negative integers. For ODE it has been proved by Yoshida (1983) that if any of the Kowalevskaya exponents (KE) are imaginary or irrational, the system is algebraically non-integrable. The KE obtained by Yoshida's singular point analysis (Yoshida 1983) have the same value as that of resonances under suitable conditions. The general connection between KE and resonances has been discussed by Roekaerts and Schwarz (1987) and further by Joy and Sabir (1988).

In the third step we test whether positive resonances do indeed correspond to arbitrary constants in the solution (12) for the full equations of motion, without logarithmic singularities. This is done by expanding the solution (12) up to the largest value of the resonance. At each resonance we come across certain conditions on the preceding u_{ij} and ϕ, known as 'compatibility conditions' which must be satisfied in order to ensure that the corresponding u_{ij} is indeed arbitrary. If the system passes all the three steps we say that it is a P-case. (Note that here the possibility of movable essential singularities are not excluded.) This method has direct connections with Bäcklund transformations, Lax pairs, Lie symmetries, etc. The WTC method described here can also be applied to ODE. In the case of ODE if we put $\phi = t - t_0$, we have the usual ARS Painlevé test.

In the case of the YM and YMH systems we shall apply the WTC method to the PDE (1) and (7) and also to the ODE obtained from them.

3.2. *Non-integrabilitity of* YM *and* YMH *systems*

We shall now apply the WTC method to the system (1) by trying to find solutions of the form

$$K = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j \qquad u_0 \neq 0.$$

To find the leading-order behaviour we put $K = u_0 \phi^{\alpha}$. We can see that $\alpha = -1$ and $u_0^2 = 2r^2(\phi_r^2 - \phi_t^2)$. The recursion relation is

$$\begin{aligned} r^2[(j-1)(j-2)(\phi_r^2 - \phi_t^2)u_j + (j-2)u_{j-1}(\phi_{rr} - \phi_{tt}) \\ + 2(j-2)(\phi_r u_{j-1,r} - \phi_t u_{j-1,t}) + u_{j-2,rr} - u_{j-2,tt}] \\ = \sum_{n=0}^{j} \sum_{s=0}^{n} u_{j-n} u_{n-s} u_s - u_{j-2}. \end{aligned}$$

Resonances are found to be -1 and 4. The resonance -1 corresponds to the arbitrariness of ϕ. For the system to be integrable, at the resonance value 4 the expansion coefficient must be arbitrary. From the recursion relations up to $j = 4$, we can see that u_4 is not arbitrary. Therefore the system does not possess the PP. The conclusion is that spherically symmetric time-dependent Yang-Mills equations are non-integrable in the sense of WTC. To see whether it is integrable in the sense of ARS we shall do a Painlevé analysis of the ODE (3) and (5) obtained from (1). Following the steps as mentioned above, we find that even though resonances are rational, a sufficient number of arbitrary expansion coefficients does not exist and hence these systems are also non-integrable.

Next we consider the spherically symmetric time-dependent Yang-Mills-Higgs system (6). We seek solutions of the form

$$K = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j \qquad H = \phi^{\beta} \sum_{j=0}^{\infty} v_j \phi^j \qquad u_0 \neq 0 \qquad v_0 \neq 0.$$

From the leading-order analysis $\alpha = \beta = -1$,

$$u_0^2 = \left(1 - \frac{\lambda}{g^2}\right) v_0^2 \qquad \text{and} \qquad \left(2 - \frac{\lambda}{g^2}\right) v_0^2 = 2r^2(\phi_r^2 - \phi_t^2).$$

Recursion relations for u_j and v_j are

$$r^2[(j-1)(j-2)(\phi_r^2-\phi_t^2)u_j+(j-2)u_{j-1}(\phi_{rr}-\phi_{tt}) \\ +2(j-2)(\phi_r u_{j-1,r}-\phi_t u_{j-1,t})+u_{j-2,rr}-u_{j-2,tt}] \\ =\sum_{n=0}^{j}\sum_{s=0}^{n} u_{j-n}(u_{n-s}u_s-v_{n-s}v_s)-u_{j-2}$$

$$r^2[(j-1)(j-2)(\phi_r^2-\phi_t^2)v_j+(j-2)v_{j-1}(\phi_{rr}-\phi_{tt}) \\ +2(j-2)(\phi_r v_{j-1,r}-\phi_t v_{j-1,t})+v_{j-2,rr}-v_{j-2,tt}] \\ =\sum_{n=0}^{j}\sum_{s=0}^{n} v_{j-n}\left(2u_{n-s}u_s-\frac{\lambda}{g^2}v_{n-s}v_s\right)-m^2r^2v_{j-2}.$$

Resonances are found to be real if $-\frac{2}{7}\leqslant(\lambda/g^2)\leqslant 2$. But for the resonances to be integers $\lambda/g^2=0$ or 1. When $\lambda/g^2=1$, $u_0=0$ or $v_0=0$ which is not allowed by the assumption that $u_0\neq 0$, $v_0\neq 0$. The resonance values for $\lambda/g^2=0$ are -1, 1, 2 and 4. Arbitrary expansion coefficients do not exist at the resonance values. Hence the system is non-integrable. When $\lambda/g^2=0$ the leading-order terms of the system (6) are equal to its PS limit (7). It is also of non-Painlevé type, and hence non-integrable. The reduced systems (8) and (9) of the PS limit are also found to be non-integrable by the same analysis.

It may also be mentioned that the ARS method is not suitable for equations (3) and (8) but can be applied to (5) and (9) after they are converted to corresponding autonomous systems. In these cases we find that the resonances, which also happen to be the KE, are irrational and hence these systems are also algebraically non-integrable in the sense of Yoshida (1983).

4. Summary and conclusions

In this work we showed that spherically symmetric time-dependent Yang-Mills equations as well as Yang-Mills-Higgs equations do not possess the PP in the sense of WTC and the ODE obtained from them are algebraically non-integrable. These conclusions are in general agreement with those obtained by Matinyan *et al* (1986, 1988) and by Furusawa (1987) for SU(2) Yang-Mills system. The noteworthy point is that we have been able to arrive at these results without introducing discretisation at any stage.

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References

Ablowitz M J, Ramani A and Segur H 1980 *J. Math. Phys.* **21** 715
Arodz H 1983 *Phys. Rev.* D **27** 1903

Babu Joseph K and Baby B V 1985 *J. Math. Phys.* **26** 2746
Fermi E, Pasta J R and Ulam S 1955 Studies of Nonlinear Problems *Report* Los Alamos LA-1940
Furusawa T 1987 *Nucl. Phys.* B **290** 469
Gorski A 1984 *Acta Phys. Polon.* B **215** 465
Jimbo M, Kruskal M D and Miwa T 1982 *Phys. Lett.* **92A** 59
Joy M P and Sabir M 1988 *J. Phys. A: Math. Gen.* **21** 2291
Matinyan S G, Prokhorenko and Savvidy G K 1986 *JETP Lett.* **44** 138
—— 1988 *Nucl. Phys.* B **298** 414
Matinyan S G, Savvidy G K and Ter Arutyunyan-Savvidy 1981 *Sov. Phys.-JETP* **53** 421
Mecklenberg W and O'Brein D P 1978 *Phys. Rev.* D **18** 1327
Nikolaevskii E S and Schur L N 1982 *JETP Lett.* **36** 218
—— 1983 *Sov. Phys.-JETP* **58** 1
Olsen P 1982 *Nucl. Phys.* B **200** 381
Ramani A, Dorizzi B and Grammaticos B 1982 *Phys. Rev. Lett.* **49** 1539
Roekaerts D and Schwarz F 1987 *J. Phys. A: Math. Gen.* **20** L127
Savvidy G K 1984 *Nucl. Phys.* B **246** 302
Steeb W-H, Louw J A and Villet C M 1986 *Phys. Rev.* D **33** 1174
Villarroel J 1988 *J. Math. Phys.* **29** 2132
Ward R S 1984 *Phys. Lett.* **102A** 279
Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522
Yoshida H 1983 *Celest. Mech.* **31** 363, 381